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THE MOTION OF A PLATE IN A MELTING SOLID MEDIUM*

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The motion of a semi-infinite heated flat plate through a solid medium, with the deformation of a melt layer at the plate surface, is considered. The flow of the melt is determined in the thin-layer approximation taking into account inertial terms in the equation of motion and a dissipative term in the heat equation. A procedure for determining an exact selfsimilar solution is described and an asymptotic method is developed for the approximate representation of the solution in terms of simple formulae. Simple estimating formulae are obtained for the length of the fluid cavity behind a plate of finite length.

A solid completely or partly immersed in a solid medium may move in the medium under the action of applied forces or by inertia, if the temperature of the body exceeds the melting (or vaporization) point of the solid medium; the body will then move together with the liquid or gas cavity formed in its vicinity. At the leading edge of the cavity the solid medium melts (vaporizes) and at the trailing edge it resolidifies. The cavity may have a free boundary or extend to infinity in the medium behind the body. Examples of such motions are illustrated in Figs.1 and 2.

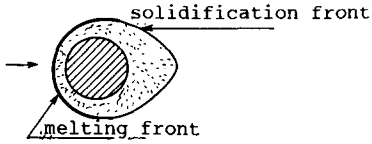

Fig.1

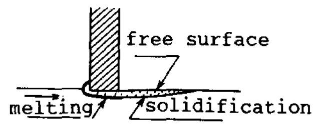

Fig.2

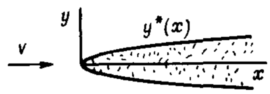

Fig.3

Since the fluid inside the cavity is in motion, the source of the heat necessary to melt or vaporize the medium will be not only the heated body but also the liquid, in which heat is generated by viscous dissipation of part of the mechanical energy delivered to the fluid by the body. Under certain conditions of motion (if the body is moving at a high velocity relative to the solid medium or if the liquid or gaseous layer is thin), the amount of heat produced by viscous dissipation may equal or even exceed the amount necessary to melt or vaporize the medium. Under such conditions the total heat flux emitted by the body may vanish or even become negative; the heat generated in the cavity will be delivered in that case to both the solid medium and the moving body.

Some problems involving the motion of heated solids in a melting medium have been considered previously in the theoretical literature (/1-3/ and elsewhere). These studies have been devoted basically to slow motion, under conditions in which inertial effects in the liquid phase ("creeping" flow) can be ignored and no allowance need be made for heat produced by viscous dissipation. Neither have any estimates been made as to the length of the liquid- or gas-filled phase of the wake behind a moving hot body of finite size.

In recent years various technological applications, as well as the discovery of new physical phenomena, have aroused interest in motion under conditions in which inertial effects and, in particular, viscous dissipation may no longer be ignored. In /6/ we solved the problem of a semi-infinite heated plate moving in a melting or vaporizing medium taking both of the above-mentioned effects into account. In the present paper** we continue the

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work of /6/, applying techniques that may form the basis for a general asymptotic approach to the problem involving the motion of solids with close-contact melting.

Let us consider a semi-infinite flat plate of zero thickness moving in its own plane, at constant velocity V, in an unbounded homogenous solid medium whose temperature at infinity is T_∞ (Fig.3); it is assumed that the plate has the same temperature T_w along its entire length, and that this temperature exceeds the melting point T_m of the solid medium ($T_m \geqslant T_\infty$). Near the plate a melt layer forms, of density ρ, viscosity μ, thermal conductivity λ and specific heat c. For simplicity, we shall assume that all these quantities, as well as the parallel quantities ρ_s, λ_s and c_s for the solid medium, are constants, independent of the pressure p and temperature T. The latent heat h_f absorbed when the solid medium melts and the temperature T_m are also assumed to be constant.

Whenever we are considering the motion of the liquid in the layer, we shall assume that the deformation of the solid medium may be neglected (if necessary, the stress-strain state of the medium may be determined from the stress distribution at the solid-liquid interface as determined by solving the flow problem; it will follow from the sequel that under certain conditions, if the velocity of motion is high or the melt layer is very thin, the loads applied to the solid-liquid interface may be significant).

It will be convenient to consider the plate as stationary, with the solid medium performing translational motion along the plate.

Assuming that the later is sufficiently thin, the motion of the melt may be described by the usual equations of motion in a thin layer /7/:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{dp}{dx}+\mu\frac{\partial^2 u}{\partial y^2},\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$
$$\rho c\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\lambda\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

Here u and v are the velocity components along and perpendicular to the plate and x and y are the coordinates in these directions, with the origin at the leading edge of the plate. Eqs.(1) are not applicable in the neighbourhood of the leading edge of the plate, where we cannot assume that the velocity components u and v are of different orders of magnitude ($v \ll u$).

Unlike the well-known equations of boundary-layer flow for bodies immersed in liquids, the function dp/dx in the first equation of (1) is not known but has to be determined, as it is in the hydrodynamical theory of lubrication /7/. Clearly, with the usual assumption that the density ρ is a constant, the pressure $p(x)$ in the layer is determined, apart from a constant. (The same is true of the motion of bodies of a more general shape, when one is determining the pressure distribution in a melt-filled closed cavity).

We shall assume that, as in the third equation of (1), the temperature distribution T_s in the solid medium may be determined ignoring the heat flux in the direction along the plate. The equation will then be

$$\rho_s c_s V\frac{\partial T_s}{\partial x}=\lambda_s\frac{\partial^2 T_s}{\partial y^2} \tag{2}$$

Strictly speaking, this equation, too, only holds in a thin layer of the solid medium, in the immediate vicinity of the melt layer. Outside this layer the "inner" solution of Eq.(2) must be matched with the "outer" solution of the complete heat equation in the moving medium. Alternatively, Eq.(2) may be considered to hold if the solid medium is thermally anisotropic and conducts heat only crosswise to the plate.

The system of four Eqs.(1) and (2) is of seventh order. Apart from the quantities which depend on x and y, i.e., u, v, T, T_s, the pressure distribution $p(x)$ and solid-liquid interface $y^*(x)$ must also be determined. Thus, in order to solve the problem we need nine boundary conditions. These will now be listed.

On the plate, $y=0,\ x\geqslant 0$:

$$u=0,\ v=0,\ T=T_w \tag{3}$$

At the solid-liquid interface, $y=y^*(x),\ x\geqslant 0$ /8/:

$$\rho_s V dy^* = \rho u dy - \rho v dx \text{ (the equation of continuity)} \tag{4}$$

$$\lambda_s\frac{\partial T_s}{\partial y}dx=\lambda\frac{\partial T}{\partial y}dx+\rho_s V h_f dy^* \text{ (the energy equation)} \tag{5}$$

$$u=V,\ T=T_s=T_m \tag{6}$$

It is not necessary to use the momentum equation at the solid-liquid interface in this formulation of the problem of flow in the layer; that equation determines the stress components in the solid medium at the interface $y = y^*(x)$: $\sigma_{yy} = p$, $\sigma_{xy} = -\mu\partial u/\partial y$.

At infinity $y \to \infty$:

$$T_s = T_\infty \tag{7}$$

All these boundary conditions are exact, except for $u = V$ and the formulae for the stresses σ_{yy} and σ_{xy} at $y = y^*(x)$, which are accurate to the same order as Eq.(1).

In accordance with the second equation of (1), we introduce the stream function $\psi(x, y)$ and change to new variables in Eqs.(1) and (2), conditions (3)-(7) and the expression for ψ:

$$u = Vu', \quad v = \frac{V}{\sqrt{\mathrm{Re}}}v', \quad p = \rho V^2 p', \quad T = T_m T', \quad T_s = T_m T_s' \tag{8}$$
$$x = Lx', \quad y = \frac{L}{\sqrt{\mathrm{Re}}}y'$$

Here L is a quantity with the dimensions of length and $\mathrm{Re} = \rho VL/\mu$ is the Reynolds number. Omitting the primes, we obtain the transformed system of equations:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 T}{\partial y^2} + m^2\left(\frac{\partial u}{\partial y}\right)^2 \tag{10}$$
$$\frac{\partial T_s}{\partial x} = a^2\frac{\partial^2 T_s}{\partial y^2}$$

and boundary conditions

$$y = 0, \; x \geqslant 0, \; u = 0, \; v = 0, \; T = T_w \tag{11}$$
$$y = y^*, \; x \geqslant 0, \; \rho_s dy^* = \rho\,(dy^* - v dx)$$
$$\frac{\lambda_s}{\lambda}\frac{\partial T_s}{\partial y}dx = \frac{\partial T}{\partial y}dx + \varkappa dy^*, \quad u = 1, \quad T = T_s = 1$$
$$y = \infty, \; T_s = T_\infty$$

Here

$$\mathrm{Pr} = \frac{c\mu}{\lambda}, \quad m^2 = \frac{V^2}{cT_m}, \quad \varkappa = \frac{\rho_s}{\rho}\,\mathrm{Pr}\,\frac{h_f}{cT_m}, \quad a^2 = \frac{\lambda_s}{\rho_s c_s}\frac{\rho}{\mu}$$

(Pr is the Prandtl number of the liquid in the layer).

The fact that the parameter a^2 in the heat equation for the solid medium (10) incorporates the characteristics ρ and μ of the liquid phase is due to the uniform choice of new independent variables for all the equations, as in the last two formulae of (8).

The stream function $\psi(x, y)$ and its non-dimensional form $\psi'(x', y')$ are related by the formula

$$\psi = \sqrt{\nu VL}\,\psi'$$

($\nu = \mu/\rho$ is the coefficient of kinematic viscosity).

Using the same reasoning as in the solution of Blasius's problem of the boundary layer at a semi-infinite plate in a uniform viscous flow /9/, one proves that the solution u, v, T, T_s of Eqs.(1) and (2) satisfying the boundary conditions (3)-(7) is selfsimilar and depends on the natural variable $y\sqrt{V/(\nu x)}$; the functions $y^*(x)$ and $p(x)$ are then determined, apart from a constant factor.

Suppose that a solution of Eqs.(9) satisfying conditions (11) has been found. Then the corresponding solution of system (1), (2) (in the original variables) will be

$$\psi = \sqrt{\nu VL}\,\psi'\left(\frac{x}{L}, \; y\sqrt{\frac{V}{\nu L}}\right), \quad T = T_m T', \quad T_s = T_m T_s'$$

where T' and T_s' depend on the same arguments as ψ' (the dependence on constant non-dimensional parameters is not indicated), and

$$y^* = \sqrt{\frac{\nu L}{V}}\,y^{*\prime}\left(\frac{x}{L}\right), \quad p = \rho V^2 p'\left(\frac{x}{L}\right) + \mathrm{const}$$

As the new "length" variable L does not appear in the "real" physical formulation of the problem, i.e., in Eqs.(1)-(7), the solution should not depend on this parameter (the function p may contain it is an additive term). Hence it follows that the only non-dimensional variable on which the functions ψ', T' and T_s' depend is $\eta = y\sqrt{V/(\nu x)} = y'/\sqrt{x'}$. Then

$$\psi = \sqrt{\nu V x}\varphi(\eta), \; T = T_m T'(\eta), \; T_s = T_m T_s'(\eta) \tag{12}$$

For y^* and p as functions of x we obtain

$$y^* = \eta^* \sqrt{\nu x/V}, \; p = -\rho V^2 k \ln(x/L).$$

Here η^* and k are non-dimensional constants and L is an arbitrary constant with the dimensions of length.

In reality, the melting front is distant from the leading edge of the plate by an amount L_T, whose order of magnitude is given by

$$L_T \sim \lambda(T_w - T_m)/(\rho_s V h_f)$$

Thus, the selfsimilar solution only holds at distances significantly greater than L_T from the leading edge of the plate.

After substituting the stream function $\sqrt{x}\varphi(\eta)$, the functions $T(\eta), T_s(\eta)$ and expressions $y^* = \eta^*\sqrt{x}$ and $p = -k\ln x$ into Eqs.(9) and (10) and the boundary conditions (11) (once again dropping primes), we obtain the following system of ordinary differential equations and boundary conditions:

$$\varphi''' + {}^1\!/_2\varphi\varphi'' + k = 0 \tag{13}$$

$$\mathrm{Pr}^{-1} T'' + {}^1\!/_2\varphi T' + m^2\varphi''^2 = 0 \tag{14}$$

$$T_s'' + (2a^2)^{-1}\eta T_s' = 0 \tag{15}$$

$$\varphi^{(0)} = \varphi'(0) = 0, \; T(0) = T_w \tag{16}$$

$$\varphi(\eta^*) = \rho_s/\rho, \; \varphi'(\eta^*) = 1, \; T(\eta^*) = 1$$

$$T'(\eta^*) + {}^1\!/_2\varkappa\eta^* = (\lambda_s/\lambda)\, T_s'(\eta^*) \tag{17}$$

$$T_s(\eta^*) = 1, \; T_s(\infty) = T_\infty \tag{18}$$

The nine boundary conditions are sufficient to determine the seven constants of integration of Eqs.(13)-(15) and the constants η^* and k.

Eq.(15) for T_s can be integrated independently of the other two equations. Omitting the standard stages, we will merely present the solution that satisfies conditions (18):

$$T_s = 1 - (1 - T_\infty)\int_{\eta^*}^{\eta} \exp\left(-\frac{\eta^2}{4a^2}\right)d\eta \left[\int_{\eta^*}^{\infty} \exp\left(-\frac{\eta^2}{4a^2}\right)d\eta\right]^{-1}. \tag{19}$$

Hence we find the quantity $T_s'(\eta^*)$ appearing on the right of the last boundary condition (17):

$$T_s'(\eta^*) = -(1 - T_\infty)\exp\left(-\frac{\eta^{*2}}{4a^2}\right)\left[\int_{\eta^*}^{\infty} \exp\left(-\frac{\eta^2}{4a^2}\right)d\eta\right]^{-1}. \tag{20}$$

Eq.(13) is not integrable in finite form (for $k = 0$ it is identical with the well-known Blasius equation). Let $\varphi(\eta, k, \alpha)$ be any solution of Eq.(13) which satisfies the initial conditions

$$\varphi(0) = \varphi'(0) = 0, \quad \varphi''(0) = \alpha \tag{21}$$

Any such solution can be expressed in parametric form, if we put

$$\varphi = |k|^{1/4}\bar{\varphi}(|k^{1/4}|\eta, \alpha|k|^{-3/4})$$

where $\bar{\varphi}(\zeta, \beta)$ is a solution of the boundary-value problem

$$\bar{\varphi}''' + {}^1\!/_2\bar{\varphi}\bar{\varphi}'' \pm 1 = 0; \; \bar{\varphi}(0) = \bar{\varphi}'(0) = 0, \; \bar{\varphi}''(0) = \beta$$

(the plus sign if $k > 0$ and the minus sign if $k < 0$).

This notation may be useful in the numerical solution of the problem, but it offers no advantages for our analytical treatment and will therefore not be used here.

For any function $\varphi(\eta, k, \alpha)$, the solution of Eq.(14) with initial data

$$T(0) = T_w, \quad T'(0) = \tau$$

is

$$T = T_w + \int_0^\eta \exp\left(-\frac{\Pr}{2}\int_0^\eta \varphi d\eta\right)\left(\tau - m^2 \Pr \int_0^\eta \varphi''^2 \exp\left(\frac{\Pr}{2}\int_0^\eta \varphi d\eta\right) d\eta\right) \tag{22}$$

Let us denote any such solution by $T(\eta, \alpha, k, \tau)$; the dependence on the prescribed parameters T_w, Pr and m^2 is not indicated here.

These solutions φ and T satisfy conditions (16) at $\eta = 0$. Using the first three condition of (17) at $\eta = \eta^*$, we can express α, k and τ in terms of η^*. After that the last condition of (17) with $\eta = \eta^*$ (which, we recall, is the heat balance equation at the solid-liquid interface) becomes an equation from which η^* can be determined as a function of all the given parameters. The determination of η^* completes the solution of the problem.

The function $\varphi(\eta, \alpha, k)$, i.e., the solution of Eq.(13) satisfying initial conditions (21), may be expanded in series

$$\varphi = \sum_{n=0}^{\infty} \frac{A_n}{n!}\eta^n \tag{23}$$

where the first coefficients are

$$A_0 = 0,\ A_1 = 0,\ A_2 = \alpha,\ A_3 = -k,\ A_4 = 0,\ A_5 = -{}^1/_2\alpha^2$$
$$A_6 = 2\alpha k,\quad A_7 = -2k^2,\quad A_8 = \frac{11}{4}\alpha^3,\quad A_9 = -\frac{45}{3}\alpha^2 k,$$
$$A_{10} = 64\alpha k^2$$

and the others $(n \geqslant 8)$ satisfy the following recurrence relation:

$$\frac{A_{n+3}}{n!} = -\frac{1}{2}\left[\frac{A_2}{2!}\frac{A_n}{(n-2)!} + \frac{A_3}{3!}\frac{A_{n-1}}{(n-3)!} + \frac{A_5}{5!}\frac{A_{n-3}}{(n-5)!} + \dots + \frac{A_{n-3}}{(n-3)!}\frac{A_5}{3!} + \frac{A_{n-1}A_3}{(n-1)!} + \frac{A_n A_2}{n!}\right]$$

These formulae imply the following structure for series (23):

$$\varphi = \sum_{j=1}^{\infty}\left[\frac{A_{3j-1}}{(3j-1)!}\eta^{3j-1} + \frac{A_{3j}}{(3j)!}\eta^{3j} + \frac{A_{3j+1}}{(3j+1)!}\eta^{3j+1}\right]$$
$$A_{3j-1} = \sum_{l=0} c_l^{(3j-1)}\alpha^{j-4l}k^{3l},\quad A_{3j} = k\sum_{l=0} c_l^{(3j)}\alpha^{j-1-4l}k^{3l},$$
$$A_{3j+1} = k^2\sum_{l=0} c_l^{(3j+1)}\alpha^{j-2-4l}k^{3l}$$

The real coefficients c_l differ from zero only for $l = 0, 1, 2, \dots$ (and j) such that the exponent of α is non-negative.

At $k = 0$ the series for φ becomes a series long since obtained by Blasius:

$$\varphi = \sum_{j=1}^{\infty}\frac{c_0^{(3j-1)}}{(3j-1)!}\alpha^j\eta^{3j-1}$$

Thus, in specific cases, after determining the functions $\varphi(\eta, k, \alpha)$ by numerical integration of Eq.(13) or by using their series representation (23), one can determine a complete solution of the problem.

We will now exhibit an asymptotic approach to the solution of problem (13)-(18), on the assumption that η^* is a small quantity; in the process we shall also determine conditions in terms of the decisive parameters ($\Pr$, m^2, $\varkappa$, a^2, ρ_s/ρ, λ_s/λ) of the problem, under which η^* will indeed be small.

We will first write $\varphi(\eta, k, \alpha)$ as $\varphi = \eta^*\Phi(\zeta, k, \alpha, \eta^*)$, where $\zeta = \eta/\eta^*$. Eq.(13) for φ is then transformed as follows (the prime denotes differentiation with respect to ζ):

$$\Phi''' + {}^1/_2\eta^{*2}\Phi\Phi'' + k^\circ = 0 \tag{24}$$

where the new constant k° is related to the old one by $k^\circ = k\eta^{*2}$. Obviously, $\Phi''(0) = \alpha\eta^*$; put $\alpha^\circ = \alpha\eta^*$.

The conditions for determining $\Phi(\zeta, k^\circ, \alpha^\circ, \eta^*)$ will be

$$\zeta = 0,\ \Phi(0) = \Phi'(0) = 0,\ \Phi''(0) = \alpha^\circ,$$

and the conditions that determine k° and α° as functions of η^* are

$$\zeta = 1,\ \Phi(1) = N,\ \Phi'(1) = 1$$

Now expand Φ and the constants k° and α° in series in terms of the small paramater η^{*2}:

$$\Phi = \varphi_0 + \eta^{*2}\varphi_1 + \ldots,\ k^\circ = k_0 + \eta^{*2}k_1 + \ldots,\ \alpha^\circ = \alpha_0 + \eta^{*2}\alpha_1 + \ldots,$$

This yields the following equations and boundary conditions for the successive terms of the series for Φ:

$$\varphi_0''' + k_0 = 0, \quad \varphi_k''' + k_k + \frac{1}{2}\sum_{i=0}^{k-1}\varphi_i\varphi_{k-1-i}'' = 0, \quad k = 1, 2, \ldots$$

$$\varphi_0(0) = \varphi_0'(0) = 0,\ \varphi_0''(0) = \alpha_0;\ \varphi_0(1) = N,\ \varphi_0'(1) = 1\ (N = \rho_s/\rho)$$

$$\varphi_k(0) = \varphi_k'(0) = 0,\ \varphi_k''(0) = \alpha_k;\ \varphi_k(1) = \varphi_k'(1) = 0$$

It is easy to show that the functions φ_k are polynomials of degree $3 + 4k$; their coefficients are power functions of N of degree $k + 1$. The following are formulae for the first two terms of the series for Φ and for the constants α° and k°:

$$\varphi_0 = \frac{\alpha_0}{2}\zeta^2 - \frac{k_0}{6}\zeta^3, \quad \varphi_1 = \frac{\alpha_1}{2}\zeta^2 - \frac{k_1}{6}\zeta^3 - \frac{\alpha_0^2}{240}\zeta^5 + \frac{\alpha_0 k_0}{360}\zeta^6 - \frac{k_0^2}{2520}\zeta^7$$

$$\alpha_0 = 2(3N - 1), \quad \alpha_1 = -\frac{1}{60}\alpha_0^2 + \frac{1}{60}\alpha_0 k_0 - \frac{1}{315}k_0^2$$

$$k_0 = 6(2N - 1), \quad k_1 = -\frac{3}{40}\alpha_0^2 + \frac{1}{15}\alpha_0 k_0 - \frac{1}{84}k_0^2$$

Thus, in the principal approximation the velocity profile $u = \Phi'(\zeta)$ is parabolic and is associated with a certain pressure gradient, as in the case of Poiseuille-Couette flow confined between parallel plates. The fact that this gradient depends only on N shows that it is determined in the principal approximation only by the change in the density of the medium at the solid-liquid interface.

Fig.4 illustrates the velocity profile in the principal approximation:

$$u = 2(3N - 1)\, y/y^* - 3(2N - 1)(y/y_*)^2 \tag{25}$$

for a few N values. For $N = 1/2$ we get a linear velocity profile; at this value of N (in this particular approximation) the pressure gradient is zero. As N falls the pressure gradient becomes positive. For $N < 1/2$ the velocity profile becomes convex away from the flow direction; for $N < 1/3$ there is a region of back flow.

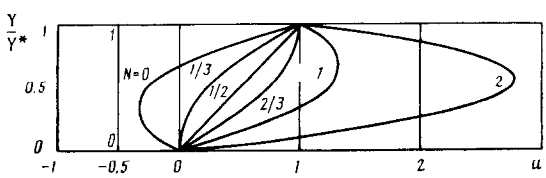

Fig.4

We emphasize that for $N > 2/3$ the maximum velocity exceeds unity and for large N values it increases rapidly; a similar rapid increase is observed in the pressure gradient. For large N values, therefore (possibly corresponding to transition into the gaseous state), allowance must be made for the compressibility of the medium in the layer.

The velocity profile (in non-dimensional variables) will depend not only on N but also on all the other parameters only in the next approximation in terms of η^{*2}, through this quantity itself, which appears as a factor multiplying the sixth-degree polynomial $\varphi_1'(\zeta)$. As to the non-dimensional pressure gradient, it too becomes dependent on all the parameters (including N) in the principal approximation through η^{*2} only, as in the simple formula

$$-x\,dp/dx = k_0/\eta^{*2} + k_1$$

Eq.(14) for the functions $T(\eta, k, \alpha, \tau, \eta^*)$ may be handled by substituting $\varphi = \eta^*\Phi(\zeta, k, \alpha, \eta^*)$, after which it becomes (the primes again denote differentiation with respect to ζ)

$$T'' + {}^1/_2\eta^{*2}\,\mathrm{Pr}\,\Phi T' + m^2\,\mathrm{Pr}\,\Phi''^2 = 0 \tag{26}$$

with the conditions

$$\zeta = 0,\ \ T = T_w,\ \ T' = \tau^\circ\ \ (\tau^\circ = \tau\eta^*)$$

To determine τ° as a function of η^*, we have the boundary condition

$$\zeta = 1,\quad T = 1$$

Expressing $T(\zeta)$ and τ° in the form

$$T = T_0 + \eta^{*2}T_1 + \ldots,\ \tau^\circ = \tau_0 + \eta^{*2}\tau_1 + \ldots$$

we obtain the following equations and initial data for the successive determination of the series terms:

$$T_0'' + m^2\,\mathrm{Pr}\,\varphi_0''^2 = 0,\quad T_k'' + \frac{\mathrm{Pr}}{2}\sum_{i=0}^{k-1}\varphi_i T'_{k-1-i} +$$

$$m^2\,\mathrm{Pr}\sum_{i=0}^{k}\varphi_i''\varphi''_{k-i} = 0,\quad k = 1, 2, \ldots$$

$$T_0(0) = T_w,\ \ T_0'(0) = \tau_0;\ \ T_k(0) = 0,\ \ T_k'(0) = \tau_k$$

Thus, the successive terms T_k of the series for T are polynomials of degree $4(k+1)$:

$$T_0 = T_w + \tau_0\zeta - m^2\,\mathrm{Pr}\left(\frac{\alpha_0^2}{2}\zeta^2 - \frac{\alpha_0 k_0}{3}\zeta^3 + \frac{k_0^2}{12}\zeta^4\right) \tag{27}$$

$$T_0' = \tau_0 - m^2\,\mathrm{Pr}\left(\alpha_0^2\zeta - \alpha_0 k_0\zeta^2 + \frac{k_0^2}{3}\zeta^3\right) \tag{28}$$

To save space, we shall not present the eighth-degree polynomial $T_1(\zeta)$ here.

The parameter τ° is found as a function of η^* from the condition $T(1) = 1$, which may be written as follows in terms of the successive terms of the series for T:

$$T_0(1) = 1,\ \ T_k(1) = 0,\ \ k = 1,\ 2,\ \ldots$$

Hence we find

$$\tau_0 = 1 - T_w + m^2\,\mathrm{Pr}\left(\frac{\alpha_0^2}{2} - \frac{\alpha_0 k_0}{3} + \frac{k_0^2}{12}\right) \tag{29}$$

The expression for τ_1 will also be omitted here.

According to (27), the temperature profile in the principal approximation is linear with respect to $\zeta\ (T = T_w + (1 - T_w)\,\zeta)$ if no allowance is made for dissipation; if dissipation is included it becomes a polynomial of degree 4 in ζ, with coefficients which depend on N, T_w and $m^2\,\mathrm{Pr}$.

The other parameters, in particular, the thermal effect of the phase change, affect the non-dimensional temperature profile only in the next approximation in terms of η^{*2}, through this parameter itself.

It is clear that the temperature distribution (27) with τ_0 determined by formula (29) is the same as in Poiseuille-Couette flow with the velocity profile (25).

Without going further with the actual solution of the problem (for the moment), we can already determine the equilibrium temperature $T_w^{(e)}$ of the plate, i.e., the temperature at which the heat flux from the plate vanishes: $T'(0) = 0$. Since $T'(0) = \tau^\circ = \tau_0 + \eta^{*2}\tau_1 + \ldots$ it follows, using the expression for τ_0 in (29), that

$$T_w^{(e)} = 1 + m^2\,\mathrm{Pr}\left(\frac{\alpha_0^2}{2} - \frac{\alpha_0 k_0}{3} + \frac{k_0^2}{12}\right) + \eta^{*2}\tau_1 + \ldots \tag{30}$$

or, confining our attention to the principal term,

$$T_w^{(e)} = 1 + m^2\,\mathrm{Pr}\,(6N^2 - 4N + 1)$$

The expression in parentheses on the right is positive and is a minimum for $N = {}^1/_3$.

If increasing m^2 (i.e., increasing velocity) makes the equilibrium temperature of the plate exceed its actual temperature, the heat flux at the plate will be directed to the plate itself and the temperature profile will show a maximum. The position of this maximum ζ_m and the temperature $T_{\max} = T(\zeta_m)$ may be determined from formulae (28) and (27) by equating the right-hand side of (28) to zero, to find ζ_m, and then substituting the result into (27). The details are extremely cumbersome (involving the solution of a third-degree equation); however, numerical analysis of specific cases is relatively easy, since the equation for ζ_m has only one real root.

To complete the solution of the problem, we determine η^* from the last condition of (17), which is the heat balance equation at the interface. Transforming the expression $T'(\eta^*)$ on the right of this condition by changing from differentiation with respect to η to differentiation with respect to ζ, we obtain

$$\frac{\lambda_s}{\lambda}\eta^* T_s'(\eta^*) = T_\zeta'(1) + \frac{\varkappa}{2}\eta^{*2}$$

where $T_s'(\eta^*)$ is the function defined by (20).

We now retain the principal term of the expansion in powers of η^*, and replace $T_s'(1)$ by the principal term of its expansion in powers of η^{*2} as obtained from (28) and (29). The result is

$$\frac{1}{a\sqrt{\pi}}\frac{\lambda_s}{\lambda}(1 - T_\infty)\eta^* + \frac{\varkappa}{2}\eta^{*2} = T_w - 1 + m^2\,\mathrm{Pr}\,(6N^2 - 8N + 3) \tag{31}$$

This equation, which expresses η^* in terms of the decisive parameters of the problem, completes the solution.

As follows from (31), when one is determining the thickness of the layer, allowance for the effect of viscous dissipation is equivalent to raising the plate temperature T_w to

$$T_w^{\mathrm{eff}} = T_w + m^2\,\mathrm{Pr}\,(6N^2 - 8N + 3)$$

(the quantity in parentheses is always positive; its minimum at $N = {}^2/_3$ is ${}^1/_3$).

Eq.(31) yields the necessary condition for η^* to be small. Solving the equation for η^*, we obtain

$$\frac{1}{a\sqrt{\pi}}\frac{\lambda_s}{\lambda}(1 - T_\infty) + \frac{\varkappa}{2} \gg T_w^{\mathrm{eff}} - 1$$

If the heat flux from the liquid layer accounts mainly for the latent heat of melting of the solid medium (the second term on the left of Eq.(31) is more significant), it must be true that

$$\frac{\varkappa}{2} \gg T_w^{\mathrm{eff}} - 1 \quad \text{or} \quad \frac{\rho_s}{2\rho}\,\mathrm{Pr}\,\frac{h_f}{c\,(T_w^{\mathrm{eff}} - T_m)} \gg 1$$

Here T_w and T_m are dimensional quantities.

If, however, the heat absorbed in melting is insignificant (the first term on the left of Eq.(31) more important), we must have

$$\frac{\lambda_s}{a\sqrt{\pi}}(T_m - T_\infty) \gg \lambda\,(T_w^{\mathrm{eff}} - T_m)$$

Here again the temperature is a dimensional quantity.

Let us determine the force X resisting the motion of the plate (i.e., of one side) over a part of length L:

$$x = \int_0^L \mu \frac{\partial u}{\partial y}\Big|_{y=0} dx = 2\mu V\sqrt{\mathrm{Re}}\,\varphi''(0) = \frac{2\rho V^2 L}{\sqrt{\mathrm{Re}}}\alpha$$

Confining our attention to the first term of the expansion for α, we get

$$X = \frac{2\rho V^2 L}{\sqrt{\mathrm{Re}}}\,\frac{\alpha_0 + \eta^{*2}\alpha_1 + \ldots}{\eta^*} = \frac{4\rho V^2 L}{\sqrt{\mathrm{Re}}}\,\frac{3N-1}{\eta^*}\left(1 + \frac{3N-2}{30}\eta^{*2}\ldots\right) \tag{32}$$

In the Blasius solution $\alpha = 0.332$, so we may write

$$X = 6.02 X_f \frac{3N-1}{\eta^*}\left(1 + \frac{3N-2}{30}\eta^{*2}\right)$$

where X_f is the resistive force for a plate immersed in an unlimited flow of viscous liquid, moving with velocity V at infinity and having the same ρ and μ as the liquid in the layer.

As we see from (32), the resistive force X may be very high at small η^* values (thus, for melting ice $\varkappa \approx 1$ and at $T_w - T_m = 2^\circ$ we obtain $\eta^* = 0.08$; in this case, therefore, $X \approx 150\, X_f$).

We shall also compute the pressure acting on the plate over a length L (assuming that $p = 0$ at $x = L$):

$$Y = \int_0^L p\,dx = -\rho V^2 L k \int_0^1 \ln x\,dx = \rho V^2 L k$$

Hence, confining our attention to the first two terms of the expansion for k, we obtain

$$Y = \frac{\rho V^2 L}{\eta^{*2}}(k_0 + \eta^{*2} k_1 + \ldots) = \frac{\rho V^2 L}{\eta^{*2}}[6(2N-1) + \eta^{*2} k_1(N) + \ldots]$$

Finally, we determine the longitudinal force acting on the solid medium (assuming that the force is positive toward negative x values):

$$X^* = \int_0^L p \frac{dy^*}{dx} dx + \int_0^L \mu \frac{\partial u}{\partial y}\bigg|_{y=y^*} dx = \frac{\rho V^2 L}{\sqrt{\mathrm{Re}}}[2k\eta^* + 2\varphi''(\eta^*)]$$

To within the accuracy of our previous considerations.

$$X^* = \frac{2\rho V^2 L}{\sqrt{\mathrm{Re}}}\frac{1}{\eta^*}\left[\alpha_0 + \eta^{*2}\left(\alpha_1 - \frac{\alpha_0^2}{12} + \frac{\alpha_0 k_0}{12} - \frac{k_0^2}{60}\right)\right]$$

Thus, in the principal approximation the forces X and X^* are of the order of $1/\eta^*$ and equal in absolute value (but opposite in direction). The difference of order η^* is due to the deviation of the momentum flow of the liquid in a cross-section of the layer at $x = L$ from the momentum flow of the solid medium near the melting front near the section.

Indeed,

$$X^* - X = \frac{\rho V^2 L}{\sqrt{\mathrm{Re}}}\eta^*\left(-\frac{\alpha_0^2}{6} + \frac{\alpha_0 k_0}{6} - \frac{k_0^2}{30}\right)$$

The difference between the momentum flows is

$$\int_0^{y^*} \rho u^2 dy - \rho_s V^2 y^* = \frac{\rho V^2 L}{\sqrt{\mathrm{Re}}}\eta^*\left(\frac{\alpha_0^2}{3} - \frac{\alpha_0 k_0}{4} + \frac{k_0^2}{20} - N\right)$$

Substituting the values of α_0 and k_0 into the expressions in parentheses in the last two formulae, we obtain two identical expressions, each equal to $^2/_5(3N^2 - 3N + {}^1/_3)$. One should note that the above difference of momenta is negative in the range of N values between the zeros of this polynomial, i.e., when $0.13 < N < 0.87$; in this range $X^* < X$.

Suppose now that the moving plate is of finite length L, which we take as our unit of length. The melt generated by the motion of the plate forms a liquid wake behind it (Fig.5), limited on both sides by solidification fronts. (At a certain distance behind the plate the solid medium may nevertheless continue to melt, because of heat transfer due to viscous dissipation in the liquid layer). At a certain $x = L_w + L$ both fronts merge (if $T_\infty < T_m$), closing the melt-filled cavity; the cavity may also extend to infinity (if $T_\infty = T_m$).

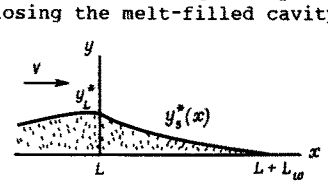

Fig.5

One should not worry about the fact that if $N \neq 1$ the mass of the melt in the cavity is not equal to the mass of solid material of the same volume, since the steady motion under consideration here may be the limit of unsteady motion, in the course of which the necessary mass balance is established (for example, if the plate enters the solid medium through a free surface); otherwise, one must allow for deformation of the solid - which is a different problem.

Remaining as before in the context of thin-layer theory, we now proceed to determine the motion of the liquid behind the plate and the shape of the cavity; to that end we must continue our solution of the system of Eqs.(9) and (10) into the region $x > L$. In so doing we must clearly replace the condition at $y = 0$ in (11) by the following condition:

$$\partial u/\partial y = 0, \ v = 0, \ \partial T/\partial y = 0$$

and consider the system with additional initial data:
at $x = 1$ (taking L as the unit of length),

$$0 < y < y_L^*, \ u = u_L(y), \ T = T_L(y); \ y > y_L^*, \ u = 1, \ T_s = T_{sL}(y) \tag{33}$$

where the functions $u_L(y)$, $T_L(y)$, $T_{sL}(y)$ are those corresponding to our previous selfsimilar distributions of the velocity component u and liquid and solid temperatures at $x = 1$. In addition, the quantity h_f appearing in the parameter $\varkappa$ is the latent heat of solidification h_s.

The solution of the non-selfsimilar problem as just formulated is quite complicated and may be achieved in specific cases by numerical methods only. We shall therefore confine ourselves to a simple approximate solution, which will enable us to give a lower limit for the length L_w of the liquid wake. To that end, we first ignore the length of the possible section of the melting front behind the trailing edge of the plate (see above). Second, of the overall heat flux that must be delivered by the liquid wake region to the solid phase, which is the sum of the superfluous heat content of the liquid flowing into the wake, the heat produced by viscous dissipation in the liquid, and the latent heat of solidification,

$$\left(\int_0^{y^*} \rho u\,[c\,(T - T_m) + h_s]\,dy\right)_{x=1} + \int_1^{L_w}\int_0^{y^*} \mu \left(\frac{\partial u}{\partial y}\right)^2 dy\,dx$$

we take into consideration only the latent heat of solidification, i.e., only the term involving h_s in the first integral. Finally, we replace the function $T_{sL}(y)$ in conditions (33) by the smaller quantity T_∞.

These three assumptions will obviously reduce the true length of the wake.

In our approximation, the temperature distribution in the solid phase and the shape of the solidification front at $x > 1$ may be found independently, without simultaneously considering the motion of the liquid. When this is done it is clear that the solution for T_s is selfsimilar and represented by the previous formula (19), with

$$\eta = (y - y_L^*)/\sqrt{x - 1} \tag{34}$$

The second condition (11) for determining η^* becomes

$$\frac{\lambda_s}{\lambda} T_s'(\eta^*) = \frac{\varkappa}{2}\eta^* \tag{35}$$

where $T_s'(\eta^*)$ is given by (20).

Solving (35) for η^* and putting $\eta = \eta^*$ in (34), we obtain the equation of the solidification surface: $y = y_L^* + \eta^* \sqrt{x - 1}$. Putting $y = 0$, we obtain the coordinate at which the solidification fronts merge, which is just the length L_w of the liquid wake:

$$\sqrt{L_w} = -y^*_L/\eta^* \tag{36}$$

We will simplify (35) for small η^* by using expression (20) for $T_s'(\eta^*)$. This gives

$$\frac{\varkappa}{2}\eta^* = -\frac{\lambda_s}{\lambda}\frac{1 - T_\infty}{a\sqrt{\pi}}$$

(that the value of η^* obtained from (35) is indeed small may be ensured by imposing the condition $c_s(T_m - T_\infty)/h_s \ll \pi/4$.)

Substituting this value of η^* into (36) we obtain a closed formula for the length of the liquid wake (in the original, dimensional variables):

$$L_w = \frac{\pi}{4}\left[\frac{h_s}{c_s(T_m - T_\infty)}\right]^2 \frac{\rho_s c_s V}{\lambda_s} y_L^{*2}$$

The coordinate y_L^* is determined by solving the problem of melt-layer formation in accordance with formula (31).

In the limiting case, ignoring heat transferred to the solid medium during melting (i.e., ignoring the first term on the left in (31)), we obtain

$$L_w = \frac{\pi}{2}\left[\frac{h_s}{c_s(T_m - T_\infty)}\right]^2 \frac{c(T_w^{\text{eff}} - T_m)}{h_f} \frac{c_s}{c} \frac{\lambda}{\lambda_s} L$$

In another limiting case, when the absorption of heat during melting may be ignored (i.e., ignoring the second term on the left in (31)), we get

$$L_w = \frac{\pi^2}{4}\left[\frac{\lambda_s}{\lambda} \frac{(T_w^{\text{eff}} - T_m)h_s}{c_s(T_m - T_\infty)^2}\right]^2 L$$

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